EE 508
Lecture 6

Degrees of Freedom

The Approximation Problem

Review from Last Time

Desgin Strategy

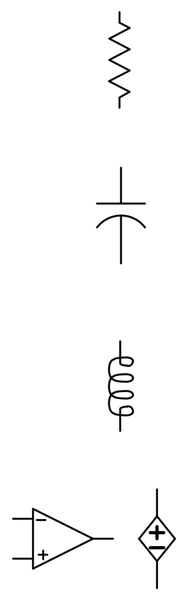
Theorem: A circuit with transfer function T(s) can be obtained from a circuit with normalized transfer function $T_n(s_n)$ by denormalizing all frequency dependent components.

$$C \rightarrow C/\omega_o$$
$$L \rightarrow L/\omega_o$$

Frequency normalization/scaling

The frequency scaled circuit can be obtained from the normalized circuit simply by scaling the frequency dependent impedances (up or down) by the scaling factor

Component denormalization by factor of ω_0



Normalized Component	Denormalized Component
R	R
C	C/ω_o
L	L/ω_o
Other Components	Unchanged

Component values of energy storage elements are scaled down by a factor of ω_0

Review from Last Time

Impedance Scaling

Theorem: If all impedances in a circuit are scaled by a constant θ, then

a) All dimensionless transfer functions are unchanged
b) All transresistance transfer functions are scaled by θ
c) All transconductance transfer functions are scaled by θ^{-1}

Impedance Scaling

Impedance scaling of a circuit is achieved by multiplying ALL impedances in the circuit by a constant

$R \rightarrow \theta R$

$C \rightarrow C/\theta$

$L \rightarrow L\theta$

$A \rightarrow$

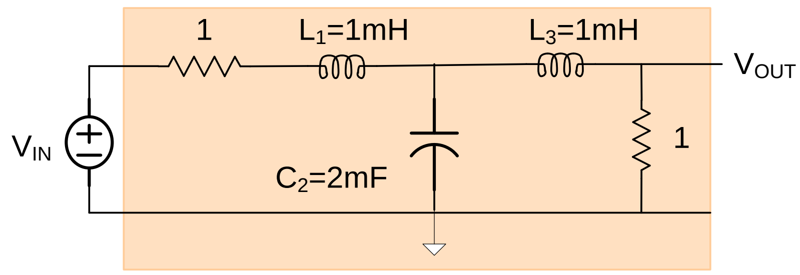
θA for transresistance gain

A for dimensionless gain

A/θ for transconductance gain

Review from Last Time

Example: Design a V-V passive 3rd-order Lowpass Butterworth filter with a band-edge of 1K Rad/Sec and equal source and load terminations.



$$T(s)=K\frac{10^9}{s^3+2\bullet10^3s^2+2\bullet10^6s+10^9}$$

Is this solution practical?

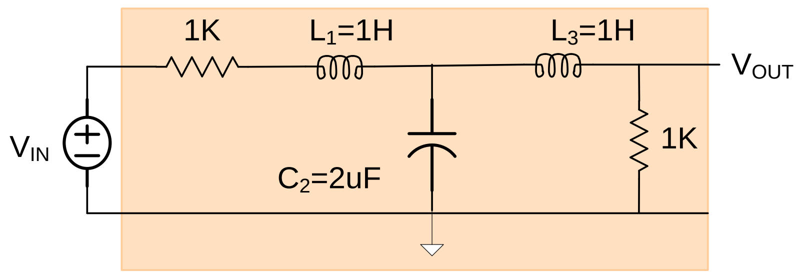
Some component values are too big and some are too small !

Impedance scale by θ=1000

R → θR

C → C/θ

L → θL



$$T(s)=K\frac{10^9}{s^3+2\bullet10^3s^2+2\bullet10^6s+10^9}$$

Component values more practical

Review from Last Time

Typical approach to lowpass filter design

1. Obtain normalized approximating function
2. Synthesize circuit to realize normalized approximating function
3. Denormalize circuit obtained in step 2
4. Impedance scale to obtain acceptable component values

Review from Last Time

Degrees of Freedom

The number of degrees of freedom in the design of a system is the difference between the total number of design variables and the number of constraints for the design.

Important to recognize the number of degrees of freedom available in a design and the number of constraints.

- If the number of design variables is less than the number of constraints in a specific system, the system is over-constrained

- Even if the number of degrees of freedom is greater than or equal to 1, a solution may not exist

Example: Design a 2nd order lowpass Butterworth filter with 3dB passband attenuation, a dc gain of 5, and a 3dB bandedge of 4KHz and no inductors

Degrees of Freedom?

Can't tell since there is no design yet

Number of Restrictions (Constraints) ?

- 2nd Order
- Lowpass
- Butterworth
- 3dB passband attenuation
- dc gain of 5
- 3dB bandedge of 4 KHz
- No inductors

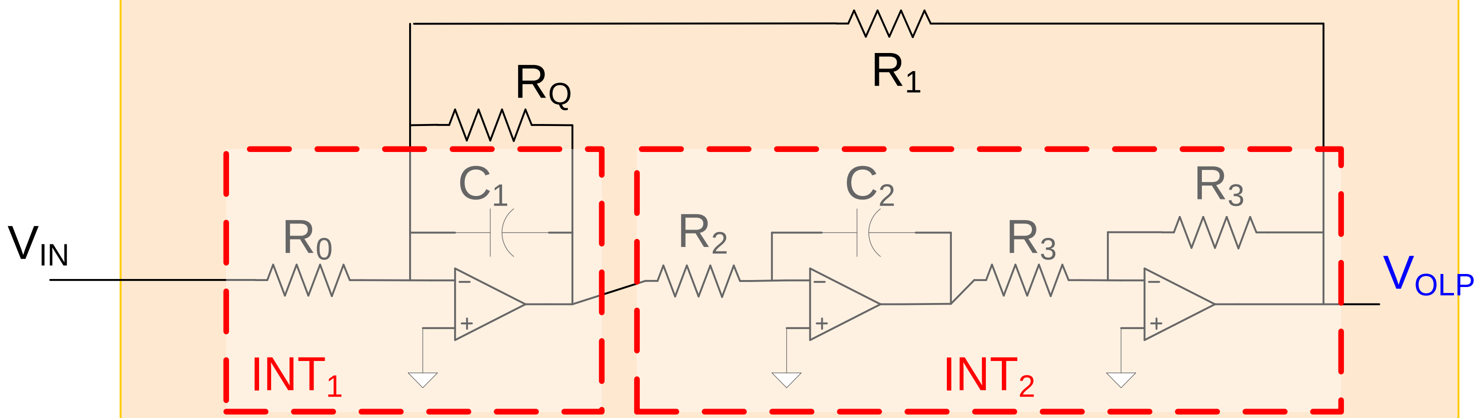
7 Restrictions

Example: Design a 2nd order lowpass Butterworth filter with 3dB passband attenuation, a dc gain of 5, and a 3dB bandedge of 4KHz with no inductors.

Note: We have not discussed the Butterworth approximation yet so some details here will be based upon concepts that will be developed later

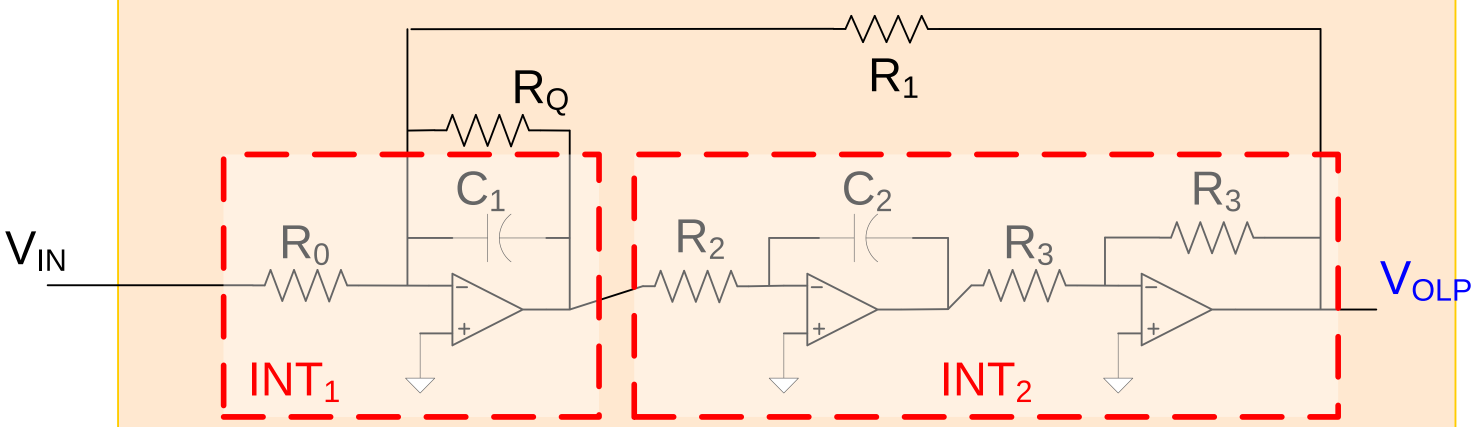
$$T_{BWn\text{-}3dB} = \left(\frac{1}{s^2+\sqrt{2}s+1}\right)\bullet 5 \quad \Longrightarrow \quad \omega_0=1,\quad Q=\frac{1}{\sqrt{2}}=0.707$$

(2nd order, lowpass, BW, 3dB, gain of 5)



A Popular Second-Order Lowpass Filter

Example: Design a 2nd order lowpass Butterworth filter with 3dB passband attenuation, a dc gain of 5, and a 3dB band edge of 4KHz,no inductors.



A Popular Second-Order Lowpass Filter

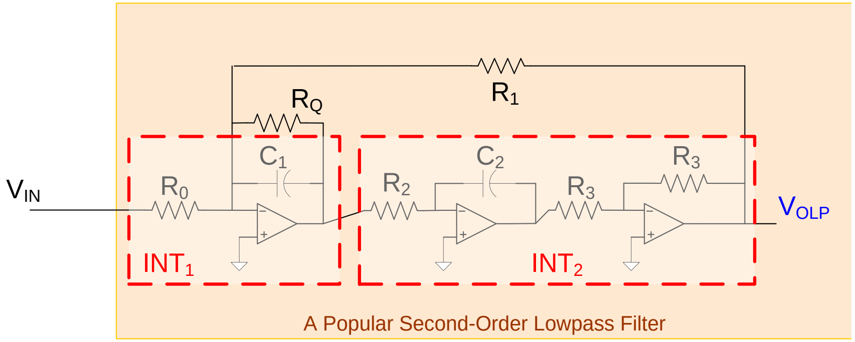
$$T(s)=\frac{\frac{1}{R_2R_0C_1C_2}}{s^2+s\left(\frac{1}{R_QC_1}\right)+\frac{1}{R_2R_1C_1C_2}}$$

$$\omega_0=\frac{1}{\sqrt{R_1R_2C_1C_2}} \qquad Q=\frac{R_Q}{\sqrt{R_1R_2}}\sqrt{\frac{C_1}{C_2}}$$

7 design variables and only two constraints (ignoring the gain right now)

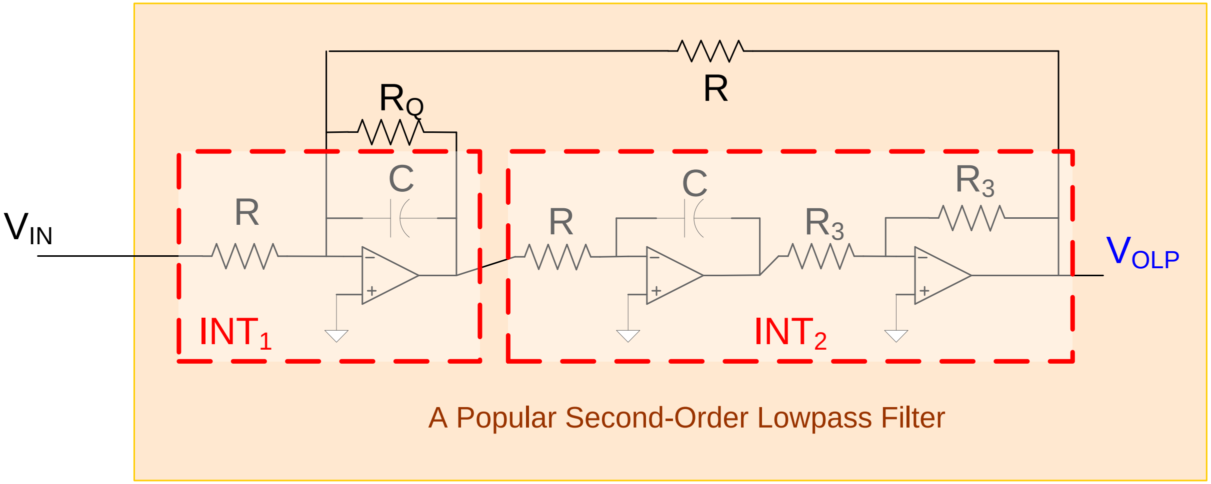
Circuit has 5 Degrees of Freedom!

Example: Design a 2nd order lowpass Butterworth filter with 3dB passband attenuation, a dc gain of 5, and a 3dB band edge of 4KHz



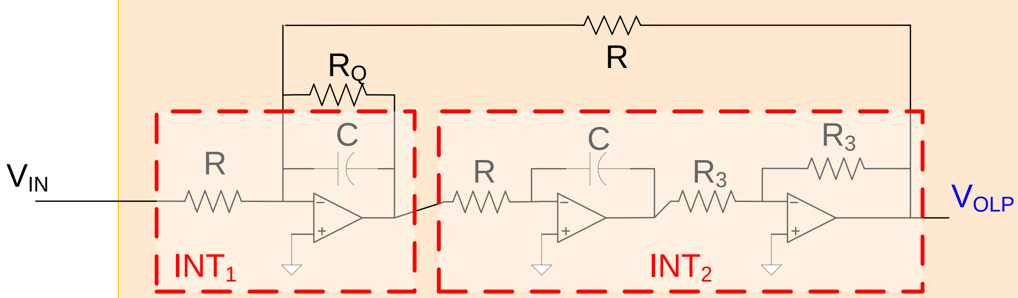
If C_1=C_2 =C and R_1=R_2=R_0 =R, this reduces to

$$T(s) = \frac{\frac{1}{(RC)^2}}{s^2 + s\left(\frac{R}{R_Q}\frac{1}{RC}\right) + \frac{1}{(RC)^2}}$$



How many degrees of freedom remain? 4-2=2

Example: Design a 2^{nd} order lowpass Butterworth filter with 3dB passband attenuation, a dc gain of 5, and a 3dB bandedge of 4KHz



A Popular Second-Order Lowpass Filter

$$T(s) = \frac{\frac{1}{(RC)^2}}{s^2+s\left(\frac{R}{R_Q}\frac{1}{RC}\right)+\frac{1}{(RC)^2}}$$

$$\omega_0 = \frac{1}{RC} \qquad Q=\frac{R_Q}{R}$$

Normalizing by the factor ω_0, we obtain

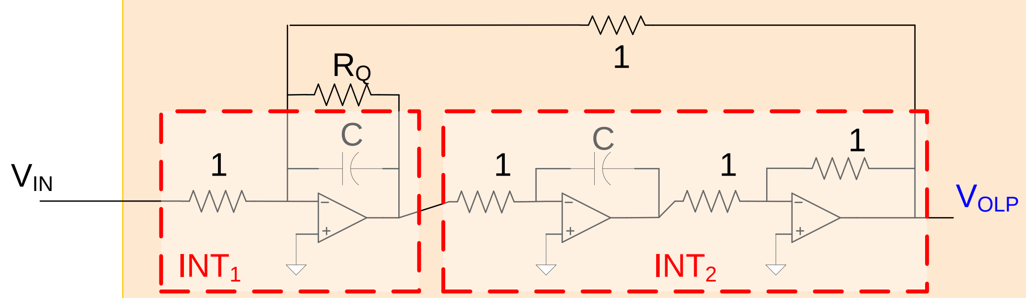
$$T(s_n) = \frac{1}{s^2+s\left(\frac{R}{R_Q}\right)+1}$$

Lets now use up the two degrees of freedom in the circuit:

Setting $R=R_3=1$ obtain the following circuit

Example: Design a 2nd order lowpass Butterworth filter with 3dB passband attenuation, a dc gain of 5, and a 3dB bandedge of 4KHz

Setting $R=R_3=1$ obtain the following circuit



A Popular Second-Order Lowpass Filter

The two constraints become

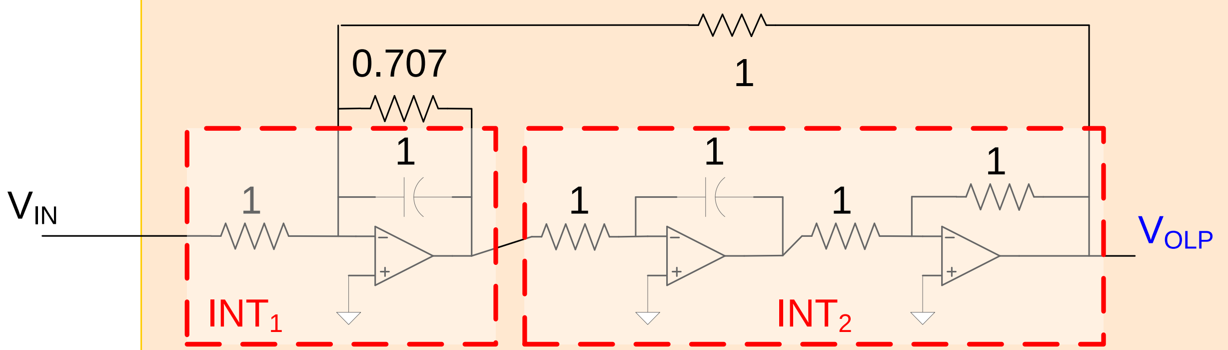
$$\omega_0 = \frac{1}{RC} = \frac{1}{C} \qquad\qquad Q = \frac{R_Q}{R} = R_Q$$

This leaves 2 unknowns, R_Q and C and two constraints (i.e. no remaining degrees of freedom)

Example: Design a 2^{nd} order lowpass Butterworth filter with 3dB passband attenuation, a dc gain of 5, and a 3dB bandedge of 4KHz

$$T(s_n) = \frac{1}{s^2 + s\left(\frac{1}{Q}\right) + 1} \qquad \omega_{0n} = 1 \qquad Q_N = \frac{1}{\sqrt{2}}$$

To satisfy the 2 constraints, must now set $R_Q = Q$ $\quad C = 1$



A Popular Second-Order Lowpass Filter

Now we can do frequency scaling

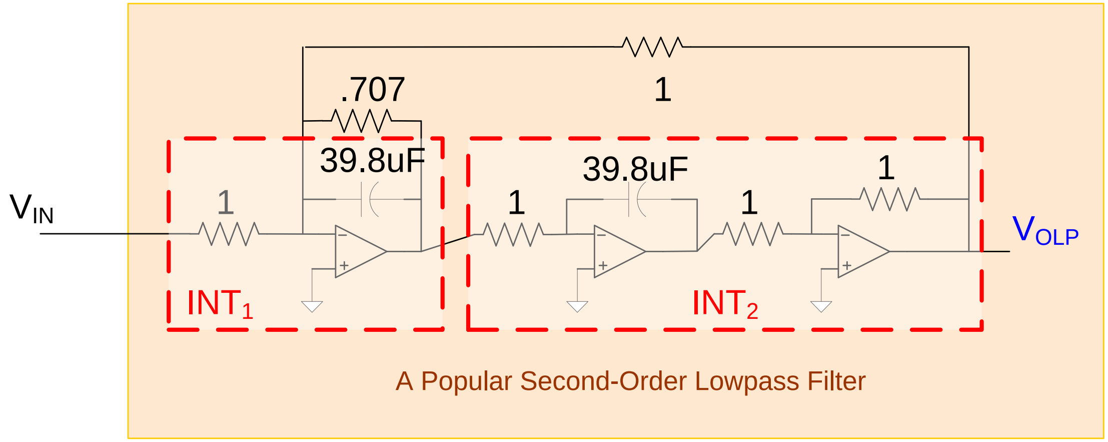
$C \rightarrow C/\omega_o$

$L \rightarrow L/\omega_o$

$C=1 \rightarrow 1/(2\pi \bullet 4K) = 39.8uF$

Example: Design a 2nd order lowpass Butterworth filter with 3dB passband attenuation, a dc gain of 5, and a 3dB bandedge of 4KHz

Denormalized circuit with bandedge of 4 KHz



This has the right transfer function (but unity gain)

Can now do impedance scaling to get more practical component values

$R \rightarrow \theta R$

$C \rightarrow C/\theta$

$L \rightarrow \theta L$

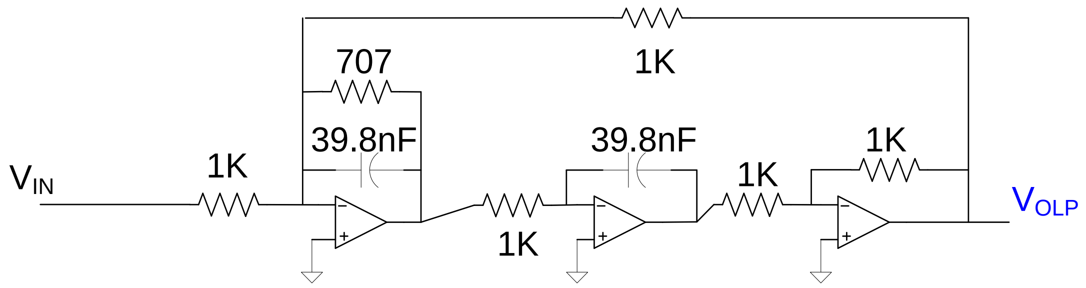
A good impedance scaling factor may be θ=1000

$R \rightarrow 1K$

$C \rightarrow 39.8nF$

Example: Design a 2^{nd} order lowpass Butterworth filter with 3dB passband attenuation, a dc gain of 5, and a 3dB bandedge of 4KHz

Denormalized circuit with bandedge of 4 KHz



This has the right transfer function (but unity gain)

To finish the design, preceed or follow this circuit with an amplifier with a gain of 5 to meet the dc gain requirements

Filter Concepts and Terminology

- Frequency scaling
- Frequency Normalization
- Impedance scaling
- Transformations
  - LP to BP
  - LP to HP
  - LP to BR

It can be shown the standard HP, BP, and BR approximations can be obtained by a frequency transformation of a standard LP approximating function

Will address the LP approximation first, and then provide details about the frequency transformations

Filter Design Process

Establish Specifications

- possibly $T_D(s)$ or $H_D(z)$
- magnitude and phase characteristics or restrictions
- time domain requirements

Approximation

- obtain acceptable transfer functions $T_A(s)$ or $H_A(z)$
- possibly acceptable realizable time-domain responses

Synthesis

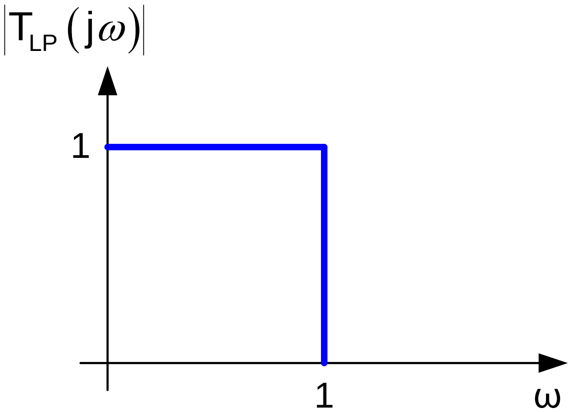
- build circuit or implement algorithm that has response close to $T_A(s)$ or $H_A(z)$
- actually realize $T_R(s)$ or $H_R(z)$

Filter

The Approximation Problem

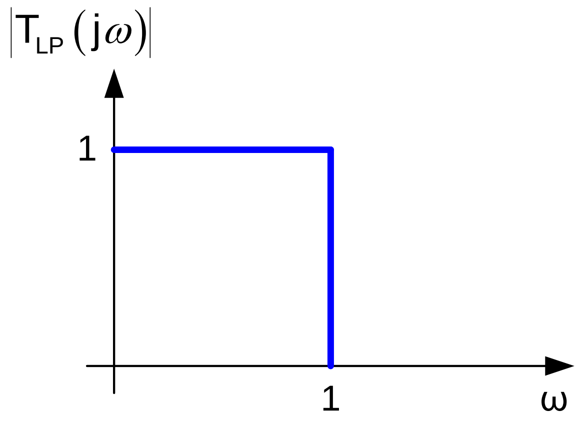
The goal in the approximation problem is simple, just want a function $T_A(s)$ or $H_A(z)$ that meets the filter requirements.

Will focus primarily on approximations of the standard normalized lowpass function



- Frequency scaling will be used to obtain other LP band edges
- Frequency transformations will be used to obtain HP, BP, and BR responses

The Approximation Problem



$$T_A(s) = ?$$

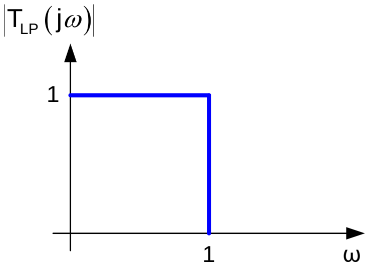
$T_A(s)$ is a rational fraction in s

$$T(s) = \frac{\sum_{i=0}^{m} a_i s^i}{\sum_{i=0}^{n} b_i s^i}$$

Rational fractions in s have no discontinuities in either magnitude or phase response

No natural metrics for $T_A(s)$ that relate to magnitude and phase characteristics (difficult to meaningfully compare $T_{A1}(s)$ and $T_{A2}(s)$)

The Approximation Problem



Approach we will follow:

- Magnitude Squared Approximating Functions $H_A(\omega^2)$
- Inverse Transform $H_A(\omega^2) \rightarrow T_A(s)$
- Collocation
- Least Squares
- Pade Approximatins
- Other Analytical Optimization
- Numerical Optimization
- Canonical Approximations
  - $\rightarrow$ Butterworth (BW)
  - $\rightarrow$ Chebyschev (CC)
  - $\rightarrow$ Elliptic
  - $\rightarrow$ Thompson

Magnitude Squared Approximating Functions

$$T(s) = \frac{\sum_{i=0}^{m} a_i s^i}{\sum_{i=0}^{n} b_i s^i}$$

$$T(j\omega) = \frac{\sum_{i=0}^{m} a_i (j\omega)^i}{\sum_{i=0}^{n} b_i (j\omega)^i}$$

$$T(j\omega) = \frac{a_o + a_1(j\omega) + a_2(j\omega)^2 + ... + a_m(j\omega)^m}{b_o + b_1(j\omega) + b_2(j\omega)^2 + ... + b_n(j\omega)^n}$$

$$T(j\omega) = \frac{\left[a_o - a_2\omega^2 + a_4\omega^4 + . .\right] + j\left[a_1\omega - a_3\omega^3 + a_5\omega^5 + ...\right]}{\left[b_o - b_2\omega^2 + b_4\omega^4 + . .\right] + j\left[b_1\omega - b_3\omega^3 + b_5\omega^5 + ...\right]}$$

$$T(j\omega) = \frac{\left[\sum_{\substack{0 \le k \le m \\ k\,\mathrm{even}}} a_k \omega^k\right] + j\left[\omega \sum_{\substack{0 \le k \le m \\ k\,\mathrm{odd}}} a_k \omega^{k-1}\right]}{\left[\sum_{\substack{0 \le k \le n \\ k\,\mathrm{even}}} b_k \omega^k\right] + j\left[\omega \sum_{\substack{0 \le k \le n \\ k\,\mathrm{odd}}} b_k \omega^{k-1}\right]}$$

$$T(j\omega) = \frac{\left[F_1(\omega^2)\right] + j\left[\omega F_2(\omega^2)\right]}{\left[F_3(\omega^2)\right] + j\left[\omega F_4(\omega^2)\right]}$$

where F_1, F_2, F_3 and F_4 are even functions of ω

Magnitude Squared Approximating Functions

$$T(s) = \frac{\sum_{i=0}^{m} a_i s^i}{\sum_{i=0}^{n} b_i s^i}$$

$$T(j\omega) = \frac{\left[F_1\left(\omega^2\right)\right] + j\left[\omega F_2\left(\omega^2\right)\right]}{\left[F_3\left(\omega^2\right)\right] + j\left[\omega F_4\left(\omega^2\right)\right]}$$

$$\left|T(j\omega)\right| = \sqrt{\frac{\left[F_1\left(\omega^2\right)\right]^2 + \omega^2\left[F_2\left(\omega^2\right)\right]^2}{\left[F_3\left(\omega^2\right)\right]^2 + \omega^2\left[F_4\left(\omega^2\right)\right]^2}}$$

Thus $\left|T(j\omega)\right|$ is an even function of ω

It follows that $\left|T(j\omega)\right|^2$ is a rational fraction in ω^2 with real coefficients

Since $\left|T(j\omega)\right|^2$ is a real variable, natural metrics exist for comparing approximating functions to $\left|T(j\omega)\right|^2$

Magnitude Squared Approximating Functions

$$T(s) = \frac{\sum_{i=0}^{m} a_i s^i}{\sum_{i=0}^{n} b_i s^i}$$

If a desired magnitude response is given, it is common to find a rational fraction in ω^2 with real coefficients, denoted as $H_A(\omega^2)$, that approximates the desired magnitude squared response and then obtain a function $T_A(s)$ that satisfies the relationship $\left|T_A(j\omega)\right|^2 = H_A(\omega^2)$

$H_A(\omega^2)$ is real so natural metrics exist for obtaining $H_A(\omega^2)$

$$H_A(\omega^2) = \frac{\sum_{i=0}^{2l} c_i \omega^{2i}}{\sum_{i=0}^{2k} d_i \omega^{2i}}$$

Obtaining $T_A(s)$ from $H_A(\omega^2)$ is termed the inverse mapping problem

But how is $T_{\underline{A}}(s)$ obtained from $H_A(\omega^2)$?

Inverse mapping problem:

$$T_A(s) \xrightarrow[\text{well defined}]{} H_A(\omega^2) \qquad H_A(\omega^2) = \left|T_A(j\omega)\right|^2$$

$$T_A(s) \xleftarrow{\ ?\ } H_A(\omega^2)$$

Consider an example:

$$T_1(s) = s+1 \rightarrow H_A(\omega^2) = 1+\omega^2$$

$$T_1(s) = s-1 \rightarrow H_A(\omega^2) = 1+\omega^2$$

Thus, the inverse mapping in this example is not unique !

Inverse mapping problem:

$$T_A(s) \longrightarrow H_A(\omega^2) \qquad\qquad H_A(\omega^2) = \left|T_A(j\omega)\right|^2$$

$$T_A(s) \xleftarrow{\color{red}{?}} H_A(\omega^2)$$

Some observations:

- If an inverse mapping exists, it is not necessarily unique

- If an inverse mapping exists, than a minimum phase inverse mapping exists and it is unique (within all-pass factors)

- The mapping from $T_A(s)$ to $H_A(\omega^2)$ increases order by a factor of 2

- Any inverse mapping from $H_A(\omega^2)$ to $T_A(s)$ will reduce order by a factor of 2 (within all-pass factors)

Example:

$$H_A\left(\omega^2\right)=\frac{2\omega^2+1}{\omega^4 + 2\omega^2 + 1} \longrightarrow T_A\left(s\right)=\frac{\sqrt{2}s+1}{\left(s+1\right)\left(s+1\right)}$$

Example:

$$H_A\left(\omega^2\right)=\frac{\omega^2-1}{\omega^4 + 2\omega^2 + 1} \longrightarrow ?$$

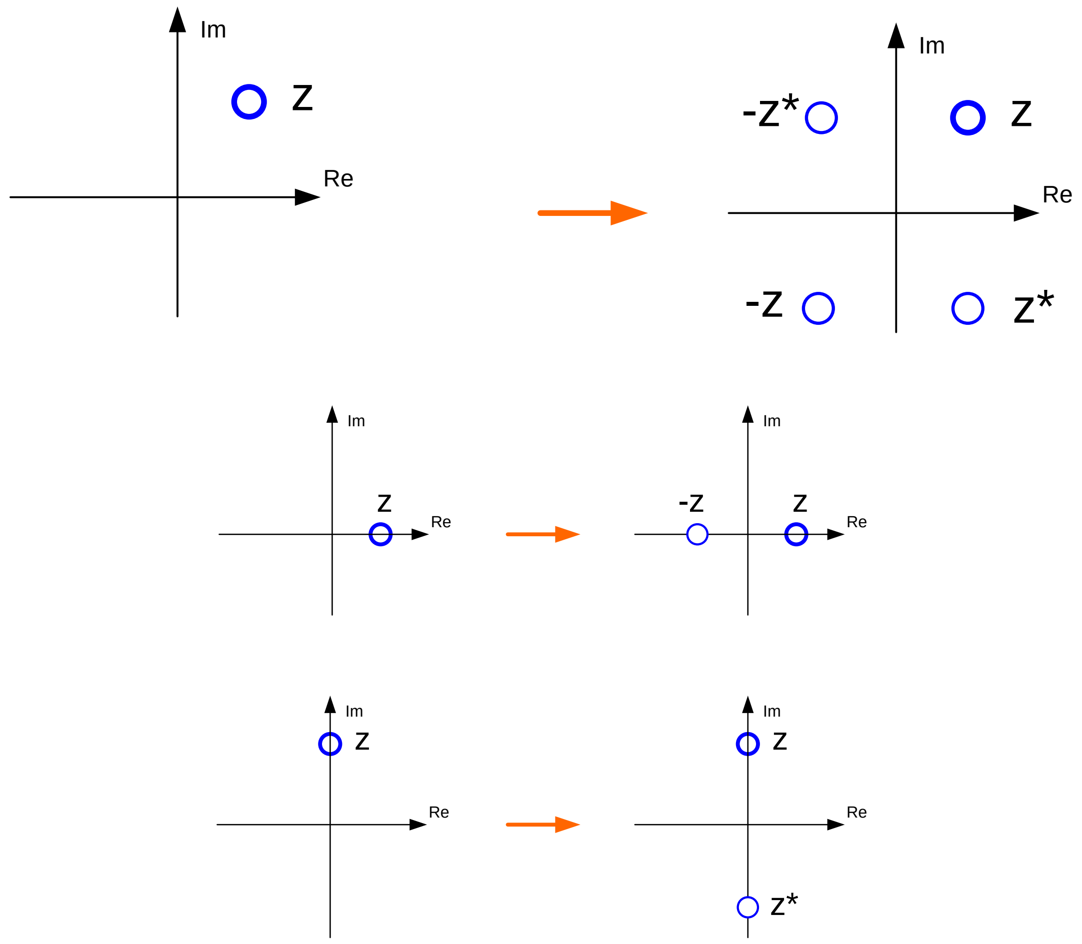
Inverse mapping does not exist !

It can be shown that many even rational fractions in ω^2 do not have an inverse mapping back to the s-domain !

Often these functions have a magnitude squared response that does a good job of approximating the desired filter magnitude response

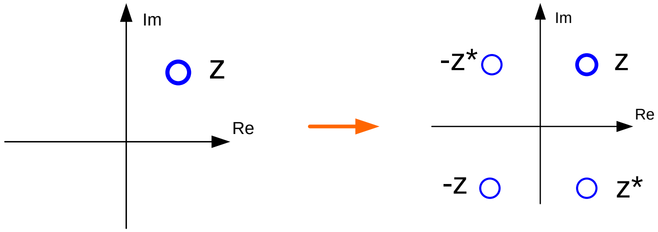
If an inverse mapping exists, there are often several inverse mappings that exist

Observation: If z is a zero (pole) of $H_A(\omega^2)$, then –z, z*, and –z* are also zeros (poles) of $H_A(\omega^2)$



Thus, roots come as quadruples if off of the axis and as pairs if they lay on the axis

Observation: If z is a zero (pole) of $H_A(\omega^2)$, then $-z$, z^*, and $-z^*$ are also zeros (poles) of $H_A(\omega^2)$



Proof:

Consider an even polynomial in ω^2 with real coefficients $P\left(\omega^2\right)=\sum_{i=0}^{m} a_i \omega^{2i}$

At a root, this polynomial satisfies the expression $P\left(\omega^2\right)=\sum_{i=0}^{m} a_i \omega^{2i}=0$

Replacing ω with $-\omega$, we obtain

$$P\left([-\omega]^2\right)=\sum_{i=0}^{m} a_i [-\omega]^{2i}=\sum_{i=0}^{m} a_i \left[-1^2\right]^i [\omega]^{2i}=\sum_{i=0}^{m} a_i [\omega]^{2i}=0 \Longrightarrow -\omega \text{ is a root of } P\left(\omega^2\right)$$

Recall $(xy)^*=x^*y^*$, $\left(x^n\right)^*=\left(x^*\right)^n$ and $(x+y)^*=x^*+y^*$

Taking the complex conjugate of $P(\omega^2)=0$ we obtain

$$P\left(\omega^2\right)^*=\sum_{i=0}^{m}\left(a_i\omega^{2i}\right)^*=\sum_{i=0}^{m}\left(a_j^*\right)\left(\omega^{2i}\right)^*=\sum_{i=0}^{m}\left(a_j^*\right)\left(\left(\omega^*\right)^{2i}\right)=0$$

Since a_i is real for all I, it thus follows that

$$\sum_{i=0}^{m}\left(a_j\right)\left(\left(\omega^*\right)^{2i}\right)=0 \Longrightarrow \omega^* \text{ is a root of } P\left(\omega^2\right)$$

Magnitude Squared Approximating Functions

If a desired magnitude response is given, it is common to find a rational fraction in ω^2 with real coefficients, denoted as $H_A(\omega^2)$, that approximates the desired magnitude squared response and then obtain a function $T_A(s)$ that satisfies the relationship $\left|T_A(j\omega)\right|^2 = H_A(\omega^2)$

Inverse mapping may not exist !

To make this approach practical it is essential that a method be developed for determining if an inverse mapping exists and, if it exists, to determine **an** inverse mapping!

Inverse MappingTheorem: If $H_A(\omega^2)$ is a rational fraction with real coefficients with no poles or zeros of odd multiplicity on the real axis, then there exists a real number H_0 such that the function

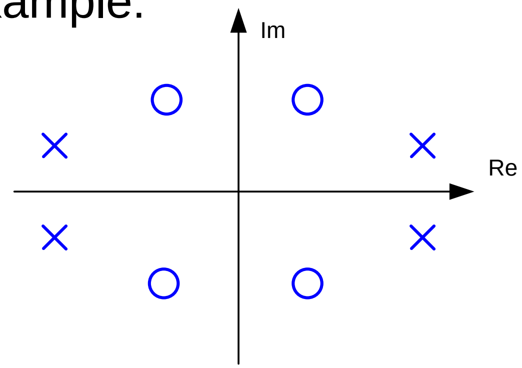
$$T_{AM}(s)=\frac{H_0(s\text{-}jz_1)(s\text{-}jz_2)\bullet\ldots\bullet(s\text{-}jz_m)}{(s\text{-}jp_1)(s\text{-}jp_2)\bullet\ldots\bullet(s\text{-}jp_n)}$$

is a minimum phase rational fraction with real coefficents that satisfies the relationship

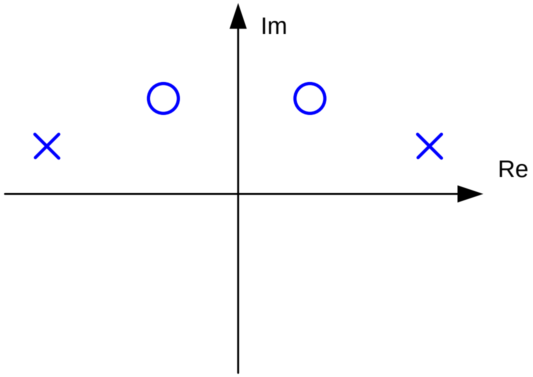
$$\left|T_{AM}(j\omega)\right|=\sqrt{H_A(\omega^2)}$$

where $\{z_1, z_2, \ldots z_m\}$ are the upper half-plane zeros of $H_A(\omega^2)$ and exactly half of the real axis zeros,
and where where $\{p_1, p_2, \ldots p_n\}$ are the upper half-plane poles of $H_A(\omega^2)$ and exactly half of the real axis poles.

Example:



Roots of $H_A(\omega^2)$



Roots that Appear in $T_{AM}(s)$

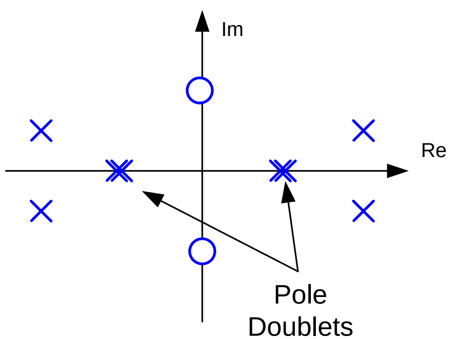
(but multiplied by j)

$$H_A\left(\omega^2\right)=\frac{H_0^2\left[(\omega-z_1)(\omega-z_2)\bullet\ldots\bullet(\omega-z_m)\right]\bullet\left[(\omega+z_1)(\omega+z_2)\bullet\ldots\bullet(\omega+z_m)\right]}{\left[(\omega-p_1)(\omega-p_2)\bullet\ldots\bullet(\omega-p_n)\right]\bullet\left[(\omega+p_1)(\omega+p_2)\bullet\ldots\bullet(\omega+p_n)\right]}$$

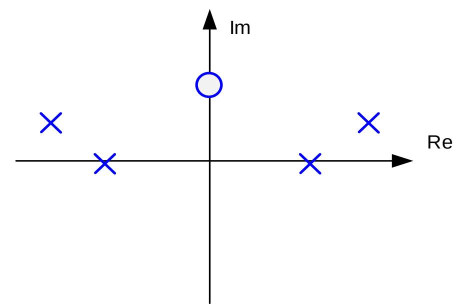
If inverse exists

$$T_{AM}(s)=\frac{H_0(s-jz_1)(s-jz_2)\bullet\ldots\bullet(s-jz_m)}{(s-jp_1)(s-jp_2)\bullet\ldots\bullet(s-jp_n)}$$

Example:

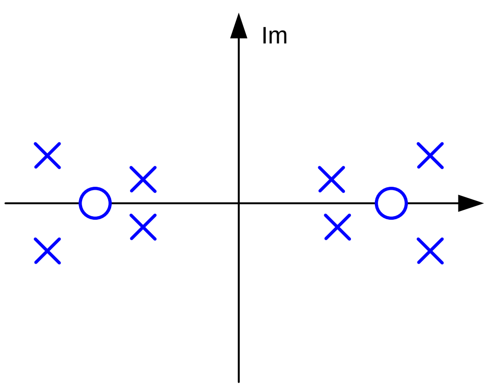


Roots of $H_A(\omega^2)$



Roots that appear in $T_{AM}(s)$

Example:

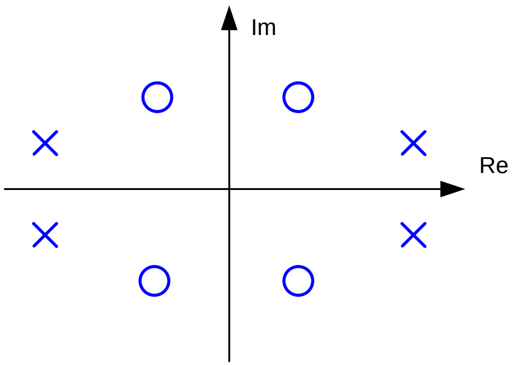


Inverse does not exist because zeros are of odd multiplicity on the real axis

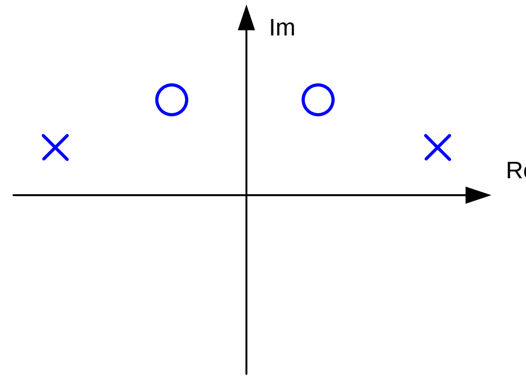
$$H_A\left(\omega^2\right)=\frac{H_0^2\left[\left(\omega-z_1\right)\left(\omega-z_2\right)\bullet\ldots\bullet\left(\omega-z_m\right)\right]\bullet\left[\left(\omega+z_1\right)\left(\omega+z_2\right)\bullet\ldots\bullet\left(\omega+z_m\right)\right]}{\left[\left(\omega-p_1\right)\left(\omega-p_2\right)\bullet\ldots\bullet\left(\omega-p_n\right)\right]\bullet\left[\left(\omega+p_1\right)\left(\omega+p_2\right)\bullet\ldots\bullet\left(\omega+p_n\right)\right]}$$

If inverse exists

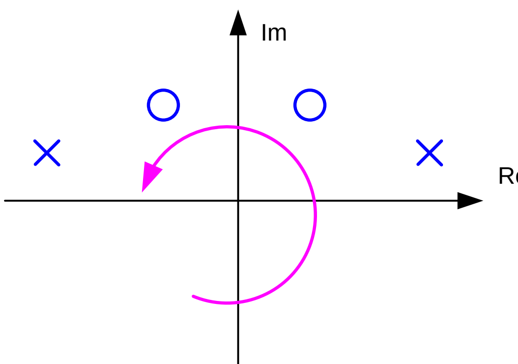
$$T_{AM}\left(s\right)=\frac{H_0\left(s-jz_1\right)\left(s-jz_2\right)\bullet\ldots\bullet\left(s-jz_m\right)}{\left(s-jp_1\right)\left(s-jp_2\right)\bullet\ldots\bullet\left(s-jp_n\right)}$$



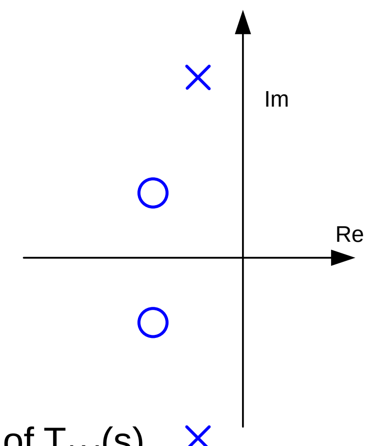
Roots of $H_A(\omega^2)$



Roots that appear in $T_{AM}(s)$



Rotate roots by 90°



Roots of $T_{AM}(s)$

Stay Safe and Stay Healthy !

End of Lecture 6